Constraint integrands for two spheres one atop the other.

By Ken Brakke, brakke@susqu.edu, November 19, 2016

This document derives the constraint content and energy integrands for a liquid bridge between two vertically stacked spheres. The two spheres are assumed to have their centers on the z axis. They need not be touching. The integrands will even work if the spheres are overlapping.

The general idea is to have each edge of a contact line responsible for a triangle of sphere surface extending to the pole. Hence it is only necessary to derive the proper integrands when the contact are is a symmetrical spherical cap. Once the cap formula is derived, 2π can be replaced by $d\theta$ to get the integrands. The signs of the integrands here assume the contact lines are oriented positively, that is, in the direction of increasing θ.

The volume and gravititational energy integrands are meant for the default mode of volume, which calculates volume vertically under a surface, and NOT for the symmetric_content mode.

Notation:

R_1	radius of lower sphere.
H_1	height of the center of the lower sphere; may be negative.
R_2	radius of the upper sphere.
H_1	height of the center of the upper sphere; may be negative, although higher than R_1.
x, y, z	coordinates of a point on the contact line (which is the rim of the spherical cap).
r_0	horizontal radius of the rim of the spherical cap. $r_0^2 = x^2 + y^2$
r	horizontal radius of a cylindrical shell, a variable of integration.
θ	circumferential angle around contact line

$$x^2 + y^2 + (z - H_1)^2 = R_1^2 \qquad r_0^2 = R_1^2 - (z - H_1)^2 \tag{1}$$

$$x^2 + y^2 + (z - H_1)^2 = R_2^2 \qquad r_0^2 = R_2^2 - (z - H_1)^2 \tag{2}$$

$$\theta = \arctan(y/x) \qquad d\theta = \frac{-y\,dx + x\,dy}{x^2 + y^2} = \frac{-y\,dx + x\,dy}{r_0^2} \tag{3}$$

Lower sphere, cap on top of sphere

Note that the forms of the integrands are not singular at the north pole, but rather at the south pole, which is not involved with the liquid (there has to be a singularity somewhere).

Lower sphere contact area, using the area of a cylinder of radius R_1 and height $H_1 + R_1 - z$:

$$Area = 2\pi(H_1 + R_1 - z)R_1 \tag{4}$$

$$dA = (H_1 + R_1 - z)R_1\,d\theta \tag{5}$$

$$= (H_1 + R_1 - z)R_1 \frac{-y\,dx + x\,dy}{r_0^2} \tag{6}$$

$$= (H_1 + R_1 - z)R_1 \frac{-y\,dx + x\,dy}{R_1^2 - (z - H_1)^2} \tag{7}$$

$$= (H_1 + R_1 - z)R_1 \frac{-y\,dx + x\,dy}{(R_1 - (z - H_1))(R_1 + (z - H_1))} \tag{8}$$

$$= R_1 \frac{-y\,dx + x\,dy}{(R_1 + z - H_1)} \tag{9}$$

Lower sphere Volume, using cylindrical shells beneath the cap. This integrand should be entered in the datafile as the negative of what is presented here, to subract off the volume below the contact surface :

$$
\begin{aligned}
Volume &= \int_{r=0}^{r_0} 2\pi r \left(H_1 + \sqrt{R_1^2 - r^2} \right) \, dr && (10) \\
&= 2\pi \int_{r=0}^{r_0} r H_1 + r\sqrt{R_1^2 - r^2} \, dr && (11) \\
&= 2\pi \left(\frac{1}{2} r^2 H_1 - \frac{1}{3}(R_1^2 - r^2)^{3/2} \right) \Bigg|_0^{r_0} && (12) \\
&= 2\pi \left(\frac{1}{2} r_0^2 H_1 + \frac{1}{3}(R_1^3 - (R_1^2 - r_0^2)^{3/2}) \right) && (13) \\
&= 2\pi \left(\frac{1}{2} r_0^2 H_1 + \frac{1}{3}(R_1^3 - (z - H_1)^3) \right) && (14) \\
dV &= \left(\frac{1}{2} r_0^2 H_1 + \frac{1}{3}(R_1^3 - (z - H_1)^3) \right) \, d\theta && (15) \\
&= \left(\frac{1}{2} r_0^2 H_1 + \frac{1}{3}(R_1^3 - (z - H_1)^3) \right) \, \frac{-y\,dx + x\,dy}{r_0^2} && (16) \\
&= \left(\frac{1}{2} H_1 + \frac{1}{3} \frac{R_1^3 - (z - H_1)^3}{r_0^2} \right) \, (-y\,dx + x\,dy) && (17) \\
&= \left(\frac{1}{2} H_1 + \frac{1}{3} \frac{R_1^3 - (z - H_1)^3}{R_1^2 - (z - H_1)^2} \right) \, (-y\,dx + x\,dy) && (18) \\
&= \left(\frac{1}{2} H_1 + \frac{1}{3} \frac{(R_1 - (z - H_1))(R_1^2 + R_1(z - H_1) + (z - H_1)^2)}{(R_1 - (z - H_1))(R_1 + (z - H_1))} \right) \, (-y\,dx + x\,dy) && (19) \\
&= \left(\frac{1}{2} H_1 + \frac{1}{3} \frac{(R_1^2 + R_1(z - H_1) + (z - H_1)^2)}{(R_1 + (z - H_1))} \right) \, (-y\,dx + x\,dy) && (20)
\end{aligned}
$$

Lower sphere Gravitational Potential Energy, using cylindrical shells. This integrand should be entered in the datafile as the negative of what is presented here, to subract off the energy below the contact surface:

$$
\begin{aligned}
U &= \rho G \int_{r=0}^{r_0} 2\pi r \left(H_1 + \sqrt{R_1^2 - r^2}\right) \frac{1}{2}\left(H_1 + \sqrt{R_1^2 - r^2}\right) dr && (21)\\
&= \pi\rho G \int_{r=0}^{r_0} r \left(H_1^2 + 2H_1\sqrt{R_1^2 - r^2} + (R_1^2 - r^2)\right) dr && (22)\\
&= \pi\rho G \int_{r=0}^{r_0} rH_1^2 + 2H_1 r\sqrt{R_1^2 - r^2} + rR_1^2 - r^3\, dr && (23)\\
&= \pi\rho G \left(\frac{1}{2}r^2H_1^2 - \frac{2}{3}H_1(R_1^2 - r^2)^{3/2} + \frac{1}{2}r^2R_1^2 - \frac{1}{4}r^4\right)\Big|_0^{r_0} && (24)\\
&= \pi\rho G \left(\frac{1}{2}r_0^2H_1^2 - \frac{2}{3}H_1(R_1^2 - r_0^2)^{3/2} + \frac{1}{2}r^2R_1^2 - \frac{1}{4}r_0^4 + \frac{2}{3}H_1R_1^3\right) && (25)\\
dU &= \frac{1}{2}\rho G \left(\frac{1}{2}r_0^2H_1^2 - \frac{2}{3}H_1(R_1^2 - r_0^2)^{3/2} + \frac{1}{2}r_0^2R_1^2 - \frac{1}{4}r_0^4 + \frac{2}{3}H_1R_1^3\right) d\theta && (26)\\
&= \frac{1}{2}\rho G \left(\frac{1}{2}r_0^2H_1^2 - \frac{2}{3}H_1(z - H1)^3 + \frac{1}{2}r_0^2R_1^2 - \frac{1}{4}r_0^4 + \frac{2}{3}H_1R_1^3\right) \frac{-y\,dx + x\,dy}{r_0^2} && (27)\\
&= \frac{1}{2}\rho G \left(\frac{1}{2}(H_1^2 + R_1)^2 - \frac{1}{4}r_0^2 - \frac{2}{3}\frac{H_1(z - H1)^3 - H_1R_1^3}{x^2 + y^2}\right) (-y\,dx + x\,dy) && (28)\\
&= \frac{1}{2}\rho G \left(\frac{1}{2}(H_1^2 + R_1)^2 - \frac{1}{4}r_0^2 - \frac{2}{3}\frac{H_1(z - H1 - R)((z - H1)^2 + (z - H1)R_1 + R_1^2)}{(R_1 - z + H_1)(R_1 + z - H_1)}\right) (-y\,dx + x\,dy) && \\
&= \frac{1}{2}\rho G \left(\frac{1}{2}(H_1^2 + R_1)^2 - \frac{1}{4}r_0^2 + \frac{2}{3}\frac{H_1((H_1 - z)^2 + (H_1 - z)R_1 + R_1^2)}{(R_1 + z - H_1)}\right) (-y\,dx + x\,dy) && (29)
\end{aligned}
$$

Upper sphere, cap on top of sphere

Note that the forms of the integrands are not singular at the south pole, but rather at the north pole, which is not involved with the liquid (there has to be a singularity somewhere).

Upper sphere contact area:

$$
\begin{aligned}
Area &= 2\pi(z - (H_2 - R_2))R_2 && (30)\\
dA &= (z - (H_2 - R_2))R_2\, d\theta && (31)\\
&= (z - (H_2 - R_2))R_2 \frac{-y\,dx + x\,dy}{r_0^2} && (32)\\
&= (z - (H_2 - R_2))R_2 \frac{-y\,dx + x\,dy}{R_2^2 - (z - H_2)^2} && (33)\\
&= (z - (H_2 - R_2))R_2 \frac{-y\,dx + x\,dy}{(R_2 - z + H_2)(R_2 + z - H_2)} && (34)\\
&= R_2 \frac{-y\,dx + x\,dy}{(R_2 - z + H_2)} && (35)
\end{aligned}
$$

Upper sphere Volume:

$$
\begin{aligned}
Volume &= \int_{r=0}^{r_0} 2\pi r \left(H_2 - \sqrt{R_2^2 - r^2} \right) \, dr & (36) \\
&= 2\pi \int_{r=0}^{r_0} r H_2 - r\sqrt{R_2^2 - r^2} \, dr & (37) \\
&= 2\pi \left(\frac{1}{2} r^2 H_2 + \frac{1}{3} (R_2^2 - r^2)^{3/2} \right) \Big|_0^{r_0} & (38) \\
&= 2\pi \left(\frac{1}{2} r_0^2 H_2 + \frac{1}{3} \left((R_2^2 - r_0^2)^{3/2} - R_2^3 \right) \right) & (39) \\
&= 2\pi \left(\frac{1}{2} r_0^2 H_2 + \frac{1}{3} \left((H_2 - z)^3 - R_2^3 \right) \right) & (40) \\
dV &= \left(\frac{1}{2} r_0^2 H_2 + \frac{1}{3} ((H_2 - z)^3 - R_2^3) \right) \, d\theta & (41) \\
&= \left(\frac{1}{2} r_0^2 H_2 + \frac{1}{3} ((H_2 - z)^3 - R_2^3) \right) \, \frac{-y \, dx + x \, dy}{r_0^2} & (42) \\
&= \left(\frac{1}{2} H_2 + \frac{1}{3} \frac{((H_2 - z)^3 - R_2^3)}{R_2^2 - (z - H_2)^2} \right) (-y \, dx + x \, dy) & (43) \\
&= \left(\frac{1}{2} H_2 + \frac{1}{3} \frac{((H_2 - z) - R_2)((H_2 - z)^2 + (H_2 - z) R_2 + R_2^2)}{(R_2 - z + H_2)(R_2 + z - H_2)} \right) (-y \, dx + x \, dy) & (44) \\
&= \left(\frac{1}{2} H_2 - \frac{1}{3} \frac{((H_2 - z)^2 + (H_2 - z) R_2 + R_2^2)}{R_2 - z + H_2} \right) (-y \, dx + x \, dy) & (45)
\end{aligned}
$$

Upper sphere Gravitational Potential Energy:

$$
\begin{aligned}
U &= \rho G \int_{r=0}^{r_0} 2\pi r \left(H_2 - \sqrt{R_2^2 - r^2} \right) \frac{1}{2} \left(H_2 - \sqrt{R_2^2 - r^2} \right) dr & (46) \\
&= \pi \rho G \int_{r=0}^{r_0} r \left(H_2^2 - 2H_2\sqrt{R_2^2 - r^2} + (R_2^2 - r^2) \right) dr & (47) \\
&= \pi \rho G \int_{r=0}^{r_0} r H_2^2 - 2H_2 r\sqrt{R_2^2 - r^2} + r R_2^2 - r^3 \, dr & (48) \\
&= \pi \rho G \left(\frac{1}{2} r^2 H_2^2 + \frac{2}{3} H_2 (R_2 - r^2)^{3/2} + \frac{1}{2} r^2 R_2^2 - \frac{1}{4} r^4 \right) \Big|_0^{r_0} & (49) \\
&= \pi \rho G \left(\frac{1}{2} r_0^2 H_2^2 + \frac{2}{3} H_2 (R_2^2 - r_0^2)^{3/2} + \frac{1}{2} r^2 R_2^2 - \frac{1}{4} r_0^4 - \frac{2}{3} H_2 R_2^3 \right) & (50) \\
dU &= \frac{1}{2} \rho G \left(\frac{1}{2} r_0^2 H_2^2 + \frac{2}{3} H_2 (R_2^2 - r_0^2)^{3/2} + \frac{1}{2} r_0^2 R_2^2 - \frac{1}{4} r_0^4 - \frac{2}{3} H_2 R_2^3 \right) d\theta & (51) \\
&= \frac{1}{2} \rho G \left(\frac{1}{2} r_0^2 H_2^2 + \frac{2}{3} H_2 (H_2 - z)^3 + \frac{1}{2} r_0^2 R_2^2 - \frac{1}{4} r_0^4 - \frac{2}{3} H_2 R_2^3 \right) \frac{-y \, dx + x \, dy}{r_0^2} & (52) \\
&= \frac{1}{2} \rho G \left(\frac{1}{2} (H_2^2 + R_2^2) - \frac{1}{4} r_0^2 + \frac{2}{3} \frac{H_2 (H_2 - z)^3 - H_2 R_2^3}{r_0^2} \right) (-y \, dx + x \, dy) & (53) \\
&= \frac{1}{2} \rho G \left(\frac{1}{2} (H_2^2 + R_2^2) - \frac{1}{4} r_0^2 + \frac{2}{3} \frac{H_2 (H_2 - z - R)((H_2 - z)^2 + (H_2 - z) R_2 + R_2^2)}{(R_2 - z + H_2)(R_2 + z - H_2)} \right) (-y \, dx + x \, dy) & \\
&= \frac{1}{2} \rho G \left(\frac{1}{2} (H_2^2 + R_2^2) - \frac{1}{4} r_0^2 - \frac{2}{3} \frac{H_2 ((H_2 - z)^2 + (H_2 - z) R_2 + R_2^2)}{(R_2 - z + H_2)} \right) (-y \, dx + x \, dy) & (54)
\end{aligned}
$$